We now turn to a particularly interesting type of systems of differential equations, known formally as a *gradient system.*

Definition. A *gradient system* on R^n is a system of differential equations of the form

$$X' = -\text{grad}V(X) \tag{1}$$

for $X \in \mathbb{R}^n$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with

$$\text{grad}V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) \tag{2}$$

The minus sign in (1) is is merely a convention, and as such we need not worry of it's significance. As displayed in (2), the gradient of a functions is simply an ordered list of it's partial derivatives. However, it's physical significance severely outweighs what it portrays to be be. The gradient of a point in a vector field is a measure of the *direction and rate of fastest increase.*

The following example is question 12, chapter 9 from Hirsch, Smale and Devaney's *Differential Equations, Dynamical Systems, and an Introduction to Chaos.*

Example 1.0. Let T be the torus defined as the square $0 \leq \theta_1, \theta_2 \leq 2\pi$ with opposite sides identified. Let $F(\theta_1, \theta_2) = \cos\theta_1 + \cos\theta_2$. Sketch the phase portrait for the system $-\text{grad}F$ in T. Sketch a three-dimensional representation of this phase portrait with T represented as the surface of the doughnut.

Let $X = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$. Then we have the following system of equations;

$$X' = \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} = \begin{pmatrix} -\partial F/\partial\theta_1 \\ -\partial F/\partial\theta_2 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \end{pmatrix} \tag{3}$$

A first order, nonlinear uncoupled system. In order to understand the way this system behaves, we must solve for the equilibrium points of the system (namely, the pairs (θ_1, θ_2) such that $X' = 0$), and then linearize the system to understand the local behaviour near these points.

The reader may notice, however, that there are in fact an infinite number of pairs (θ_1, θ_2) which solve $X' = 0$: Any pair of the form $(n\pi, m\pi)$ for $n, m \in \mathbb{Q}$ will do. To make the task easier for us, we restrict our view to the square $[0, 2\pi]_{\theta_1} \times [0, 2\pi]_{\theta_2}$. In which case, we are left with 9 distinct equilibrium which can be summarized as $\{(n\pi, m\pi) : n, m \in \{0, 1, 2\}\}$. Now that we have our equilibrium points, we want to understand the local behaviour of solutions to (3) in a neighbourhood of the equilibrium points. To do so, we take the Jacobian of X' to get a linear planar system of the form $Y' = DF_X Y$;

$$DF_X = \begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix} \tag{4}$$

If we consider the equilibrium point $(0, 0)$, we find a linear system given by

$$Y' = DF_{(0,0)}Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y \tag{5}$$

A planar system with repeated, positive eigenvalues. This is indicates that $(0, 0)$ is a source, and locally, solutions project directly outwards from this point. The identification of the other equilibrium are handled similarly, and are left to the reader. Figure 1 shows the phase portrait of the system.

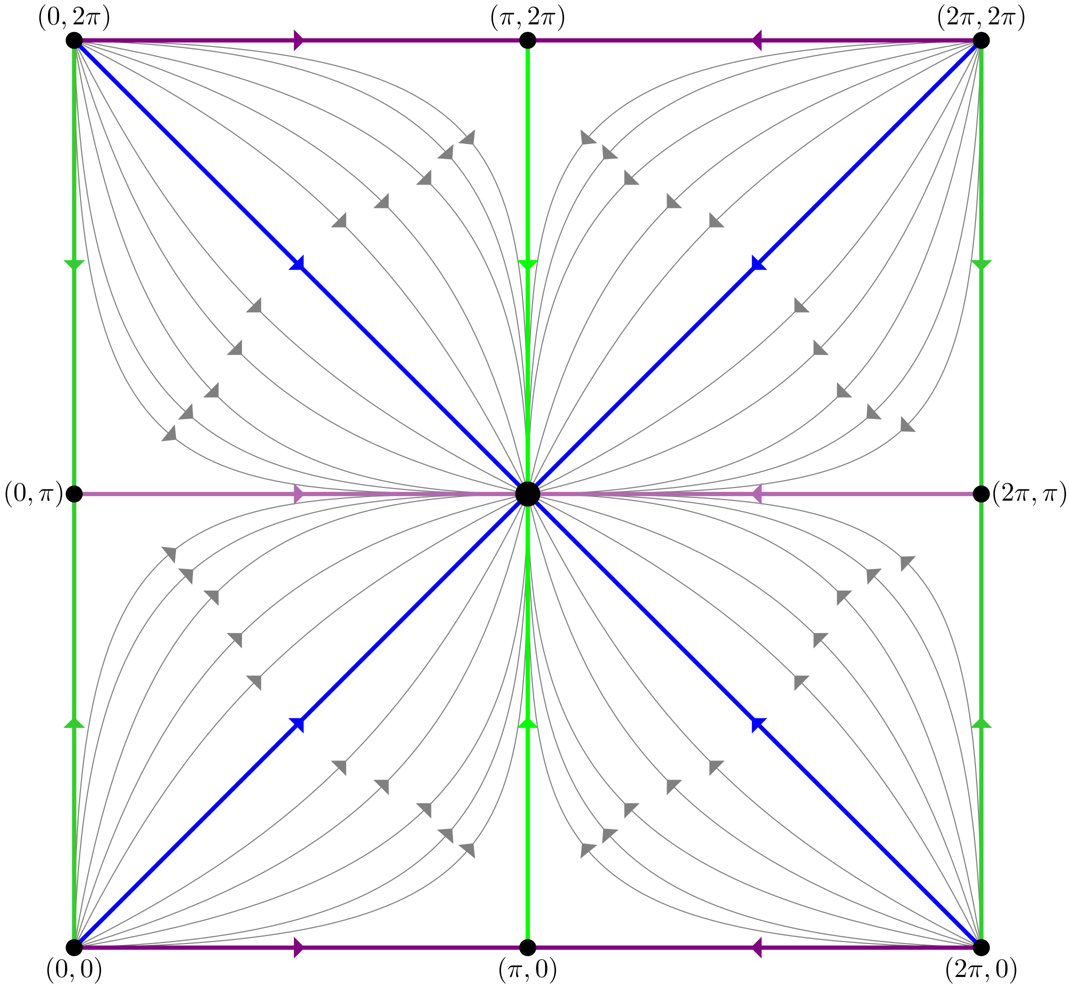

Figure 1: Phase portrait of (3) on $[0, 2\pi] \times [0, 2\pi]$

For a more interactive view of this phase portrait, visit Field Play. As is shown, at each of the four corners, solutions flow out from a source, each of which falls into the sink located at (π, π). Moreover, the midpoints of the perimeter each take the form of a saddle, with stable solutions flowing from the four corner, and the unstable lines again flowing directly towards the sink.

Moreover, we can *map* this phase portrait onto the torus. While we can certainly parameterize the surface of the torus using the map $T : [0, 2\pi]_u \times [0, 2\pi]_v \to \mathbb{R}^3$ defined by

$$T(u, v) = [(2 + \cos u) \sin u, (2 + \cos uv) \cos u, \sin u] \tag{6}$$

It is, in my opinion, far more motivating (and interesting) to visualize the topological construction of the torus. More explicitly, we will construct the torus from a rectangular subset of $\mathbb{R}^2$. See Appendix A**??**.

The phase portrait mapped onto the torus can be seen in Figure 2, with colour coordinated solutions which align with Figure 1. Only a sample of the lines have been plotted to allow for an easier viewing.

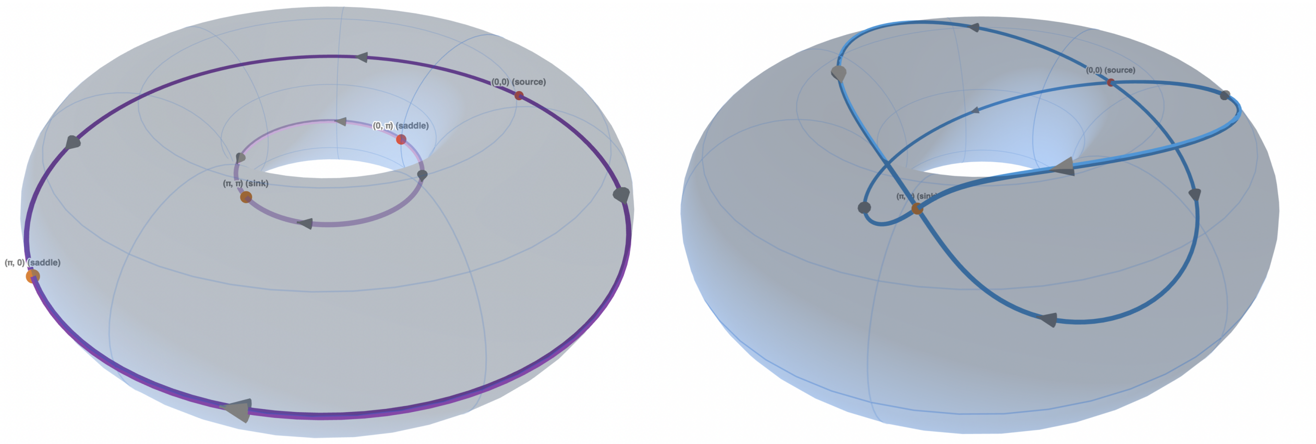

Figure 2: **a)** Purple Curves **b)** Blue Curves

For a more visual view of this torus, visit Math3D.