1. The Levin Equation

We wish to compute the following integral;

$$\int_R \exp(i\omega g(x,y)) f(x,y)\,dxdy \tag{1}$$

Where $\omega \in \mathbb{R}$ is a constant, $f, g : \mathbb{R}^2 \to \mathbb{R}$ are slowly-varying functions and $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ is a rectangle. To solve this integral, it suffices to solve the following second-order differential equation for $p : \mathbb{R}^2 \to \mathbb{R}$:

$$\frac{\partial^2}{\partial x \partial y}\left[\exp(i\omega g(x,y)) p(x,y)\right] = \exp(i\omega g(x,y)) f(x,y) \tag{2}$$

For plugging the above into Eq.(1), we find;

$$\begin{aligned}\int_a^b \int_c^d \frac{\partial^2}{\partial x \partial y}\left[\exp(i\omega g(x,y)) p(x,y)\right] dxdy &= \int_a^b \frac{\partial}{\partial x}\left[\exp(i\omega g(x,d)) p(x,d) - \exp(i\omega g(x,c)) p(x,c)\right] dx \\ &= \Big[\exp(i\omega g(b,d)) p(b,d) - \exp(i\omega g(b,c)) p(b,c)\Big] \\ &\quad - \Big[\exp(i\omega g(a,d)) p(a,d) - \exp(i\omega g(a,c)) p(a,c)\Big]\end{aligned}$$

Evaluating the LHS of Eq.(2) yields the following;

$$\begin{aligned} f(x,y)\exp(i\omega g(x,y)) &= \frac{\partial}{\partial x}\left(p_y(x,y) + i\omega g_y(x,y) p(x,y)\right)\exp(i\omega g(x,y)) \\ &= \Big\{ \left[p_y(x,y) + i\omega g_y(x,y) p(x,y)\right] i\omega g_x(x,y) \\ &\qquad + \left[p_{xy}(x,y) + i\omega\left(g_{xy}(x,y) p(x,y) + p_y(x,y) p_x(x,y)\right)\right] \Big\} \exp(i\omega g(x,y)) \\ \implies f &= p_{xy} + i\omega(g_x p_y + g_y p_x) + (i\omega g_{xy} - \omega^2 g_x g_y) p \end{aligned} \tag{3}$$

This is the Levin Equation.

2. Discretization

Introduce a 2-dimensional Chebyshev tensor product extremal grid $\{(x_i, y_j)\}_{i,j=1}^n$, and introduce the following notation:

$$[f] = \begin{pmatrix} f(x_1,y_1) \\ f(x_2,y_1) \\ \vdots \\ f(x_n,y_1) \\ f(x_1,y_2) \\ \vdots \\ f(x_n,y_n) \end{pmatrix}$$

Finally, let D denote the $n \times n$ spectral differentiation matrix. Then the $n^2 \times n^2$ matrices

$$I \otimes D \qquad\qquad D \otimes I$$

map $[f] \mapsto [f_x]$ and $[f] \mapsto [f_y]$ respectively. It follows that

$$(I \otimes D)(D \otimes I)[f] = (I \otimes D)[f_y] = [f_{xy}]$$

We now construct the following matrix $A \in M_{n^2,n^2}$:

$$\boxed{A = (I \otimes D)(D \otimes I) + i\omega\Big(\text{diag}[g_x](D \otimes I) + \text{diag}[g_y](I \otimes D)\Big) + i\omega\text{diag}[g_{xy}] - \omega^2\text{diag}[g_x g_y]}$$

It then follows that

$$\begin{aligned} A[p] &= (I \otimes D)(D \otimes I)[p] + i\omega\text{diag}[g_x](D \otimes I)[p] + i\omega\text{diag}[g_y](I \otimes D)[p] + i\omega\text{diag}[g_{xy}][p] - \omega^2\text{diag}[g_x g_y][p] \\ &= [p_{xy}] + [i\omega g_x p_y] + [i\omega g_y p_x] + [i\omega g_{xy} p] - [\omega^2 g_x g_y p] \\ &= [p_{xy} + i\omega(g_x p_y + g_y p_x) + (i\omega g_{xy} - \omega^2 g_x g_y)p] \\ &= [f] && \text{(By Eq.(3))} \end{aligned}$$