The Adaptive Levin Method

By Murdock Aubry

Supervised by Prof. James Bremer

CUMC June 2023

Numerical Integration

Why would we integrate?

- Geometric computations
  - Centre of mass
  - Moment of Inertia
  - Symmetries
  - Structural Integrity
  - Tension forces
- Trajectories of objects in motion
  - Acceleration → velocity → position
  - Rotational trajectory
- Electromagnetic computations
  - Earth's magnetic field
  - Electric field within closed system
- Financial Engineering
  - Pricing financial derivatives
  - Profit computations

Why would we *numerically* integrate?

- No Antiderivative
  - E.x. $f(x) = \exp(-x^2)$
- Only sampled points
- Easier than finding antiderivative
- Irregular or scattered data
- High dimensional integrals
- Complex domains
- Numerical verification

Numerical Integration

- Midpoint Method

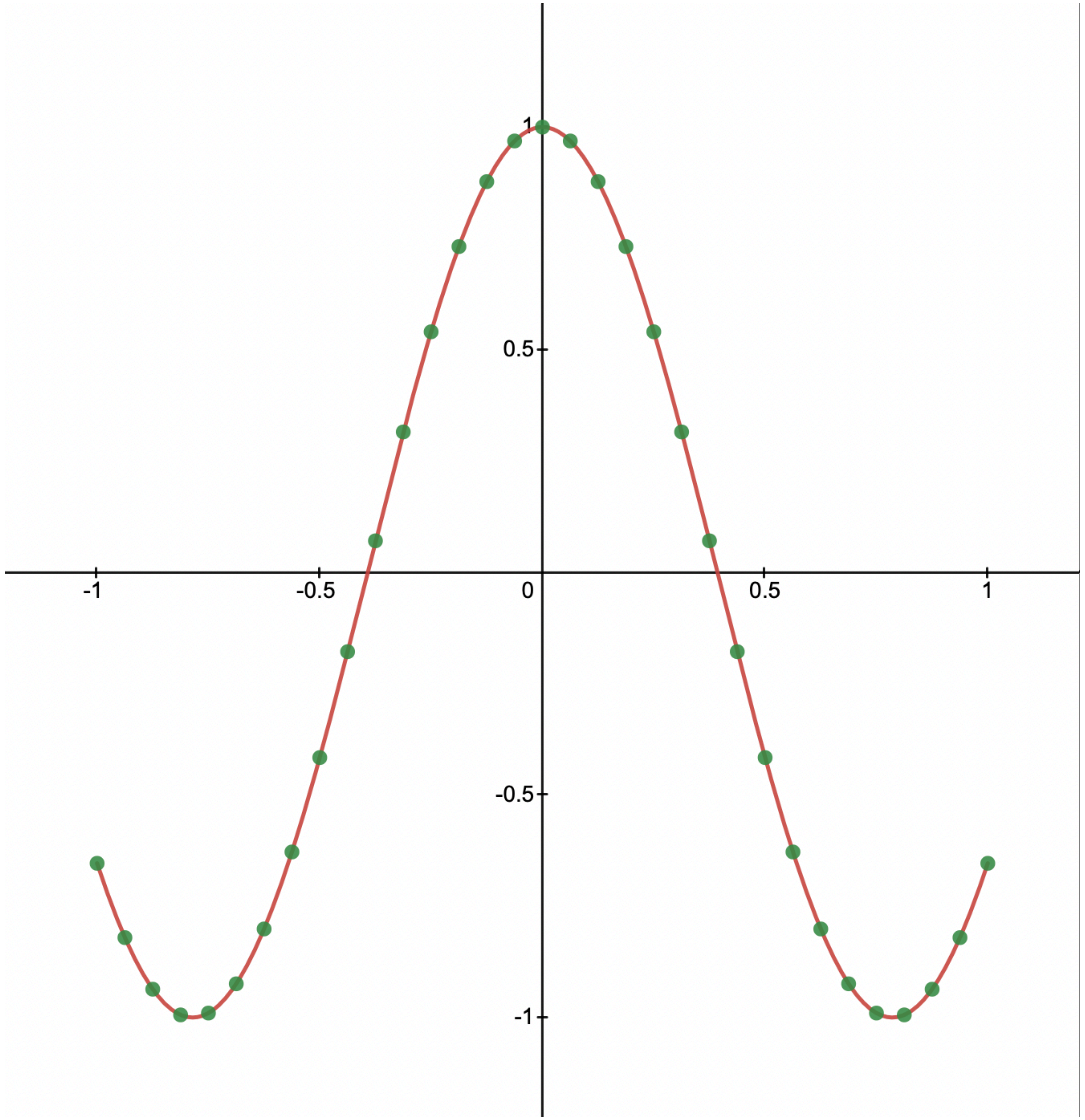
  Consider $\cos(nx)$ for $x \in [-1,1]$ and $n \in \mathbb{N}$.

  n^2 equally spaced points on $[-1,1]$.

- Gauss-Legendre Integration

  Construct Gauss-Legendre quadrature $\{x_i, w_i\}_{i=1}^{n}$

$$\int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$



One-Dimensional Levin Method

Goal: Numerically evaluate integrals of the following form;

$$I = \int_a^b f(x)\ \exp(ig(x))dx \tag{1}$$

$f, g : \mathbb{R} \to \mathbb{R}$ and $a < b \in \mathbb{R}$

$$\mathfrak{Re}(I) = \int_a^b f(x)\ \cos(g(x))dx \qquad \mathfrak{Im}(I) = \int_a^b f(x)\ \sin(g(x))dx$$

Method: Find an antiderivative of the integrand in Eq.(1).

Goal: Numerically evaluate integrals of the following form;

$$I = \int_a^b f(x)\, \exp(ig(x))dx \tag{1}$$

Search for a function $p : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)\, \exp(ig(x)) = \frac{d}{dx}\left(p(x)\, \exp(ig(x))\right)$$

Plugging this into I, we obtain

$$I = \int_a^b \frac{d}{dx}\left(p(x)\, \exp(ig(x))\right) dx = p(b)\, \exp(ig(b)) - p(a)\, \exp(ig(a))$$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\, \exp(ig(x))dx$$

Goal: Numerically evaluate integrals of the following form;

$$I = \int_a^b f(x)\, \exp(ig(x))dx \tag{1}$$

Search for a function $p : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} f(x)\, \exp(ig(x)) &= \frac{d}{dx}\left(p(x)\, \exp(ig(x))\right) \\ &= \left(p'(x) + p(x) \cdot ig'(x)\right) \exp(ig(x)) \end{aligned}$$

$$\boxed{\implies f(x) = p'(x) + p(x) \cdot ig'(x)}$$

Goal: Evaluate the following;

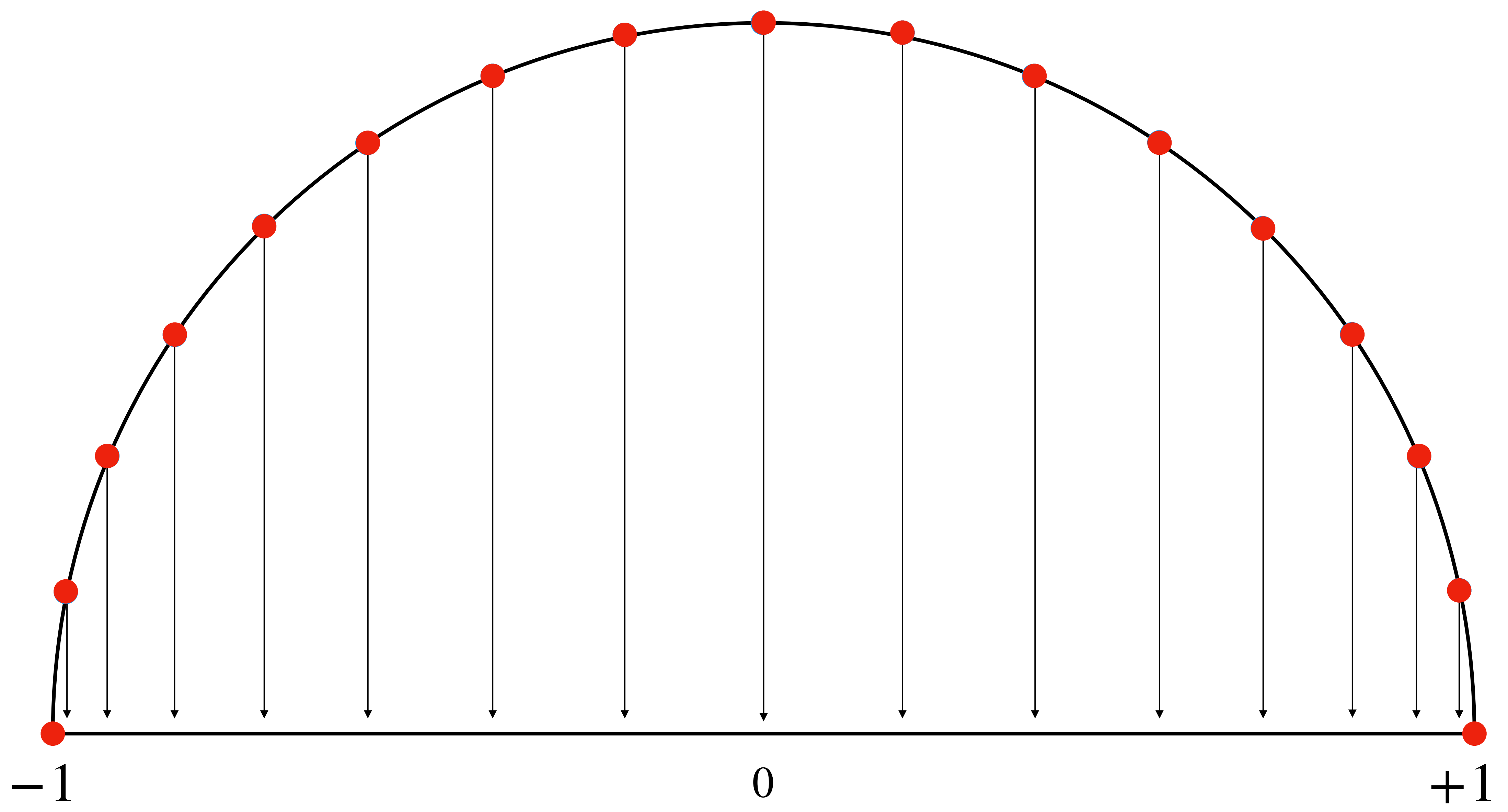
$$I = \int_a^b f(x)\, \exp(ig(x))dx$$

Method of Discretization: Chebyshev Interpolation

We define the k-point Chebyshev extremal grid on the interval $[-1,1]$ as

$$-1 = x_{1,k} < x_{2,k} < \cdots < x_{k,k} = 1$$

$$x_{j,k} = \cos\left(\pi \frac{k-j}{k-1}\right)$$



Obtain grid on interval $[a, b]$ via the map

$$L : [-1,1] \to [a, b]$$

$$L(x) = \frac{b-a}{2}x + \frac{b+a}{2}$$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\, \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Method of Discretization: Chebyshev Interpolation

Chebyshev polynomials are defined recursively;

$$T_0(x) = 1 \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad T_n(\cos\theta) = \cos(n\theta)$$

$$T_1(x) = x$$

They also satisfy the following orthogonality relation:

$$\sum_{k=0}^{n-1} T_i(x_{k,n})T_j(x_{k,n}) = \begin{cases} 0 & i \neq j \\ n & i = j = 0 \\ n/2 & i = j \neq 0 \end{cases} \qquad f(x) = \sum_{i=0}^{n-1} a_i T_i(x)$$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\ \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Method of Discretization: Chebyshev Interpolation

We can express a function f as a linear combination of these Chebyshev Polynomials, and evaluate such at the grid points $\{x_{j,k}\}$.

$$f(x) = \sum_{i=0}^{n-1} a_i T_i(x) \quad \Longleftrightarrow \quad f(x_{j,k}) = \sum_{i=0}^{n-1} a_i T_i(x_{j,k}) \qquad \forall j, k$$

Exploiting the orthogonality relation, the coefficients $\{a_i\}$ of the expansion are determined.

Moral: Given the values of a function at the Chebyshev nodes, we can interpolate the function everywhere.

Goal: Evaluate the following;

$$I = \int_a^b f(x)\, \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Method of Discretization: Chebyshev Interpolation

Let $[f]$ denote the k-vector of evaluations at the k Chebyshev nodes: $[f] = \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix}$

Denote $\mathscr{D}_k$ the $k \times k$ spectral differentiation matrix which sends $[f] \xrightarrow{\mathscr{D}_k} [f']$. That is,

$$\mathscr{D}_k \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix} = \begin{pmatrix} f'(x_{1,k}) \\ \vdots \\ f'(x_{k,k}) \end{pmatrix}$$

$\mathscr{D}_k$ simply evaluates $f'(x) = \sum_{i=0}^{n-1} a_i T_i'(x)$ at the Chebyshev nodes.

Goal: Evaluate the following;

$$I = \int_a^b f(x) \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Method of Discretization: Chebyshev Interpolation

Now consider the following matrix $\mathscr{A}$ given by;

$$\mathscr{A} = \mathscr{D}_k + i \begin{pmatrix} g'(x_{1,k}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g'(x_{k,k}) \end{pmatrix}$$

Then, applying $\mathscr{A}$ to $[p]$, we find,

$$\mathscr{A}[p] = \mathscr{D}_k \begin{pmatrix} p(x_{1,k}) \\ \vdots \\ p(x_{k,k}) \end{pmatrix} + i \begin{pmatrix} g'(x_{1,k}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g'(x_{k,k}) \end{pmatrix} \begin{pmatrix} p(x_{1,k}) \\ \vdots \\ p(x_{k,k}) \end{pmatrix} = \begin{pmatrix} p'(x_{1,k}) + ig'(x_{1,k})p(x_{1,k}) \\ \vdots \\ p'(x_{k,k}) + ig'(x_{k,k})p(x_{k,k}) \end{pmatrix} = [f]$$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\, \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Notation:

$$[f] = \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix}$$

Method of Discretization: Chebyshev Interpolation

$$\mathscr{A}[p] = [f]$$

Numerically inverting $\mathscr{A}$ via QR or SVD, we obtain:

$$[p] = \mathscr{A}^{-1}[f]$$

Given the subinterval $[a_0, b_0] \subset [a, b]$ and associated nodes $\{x_{j,k}\}$, the following approximation is made;

$$\int_{a_0}^{b_0} f(x)\ \exp(ig(x))dx \approx p(x_{k,k})\ \exp(i(g(x_{k,k})) - p(x_{1,k})\ \exp(ig(x_{1,k}))$$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\ \exp(ig(x))dx$$

First: Find $p(x)$ s.t

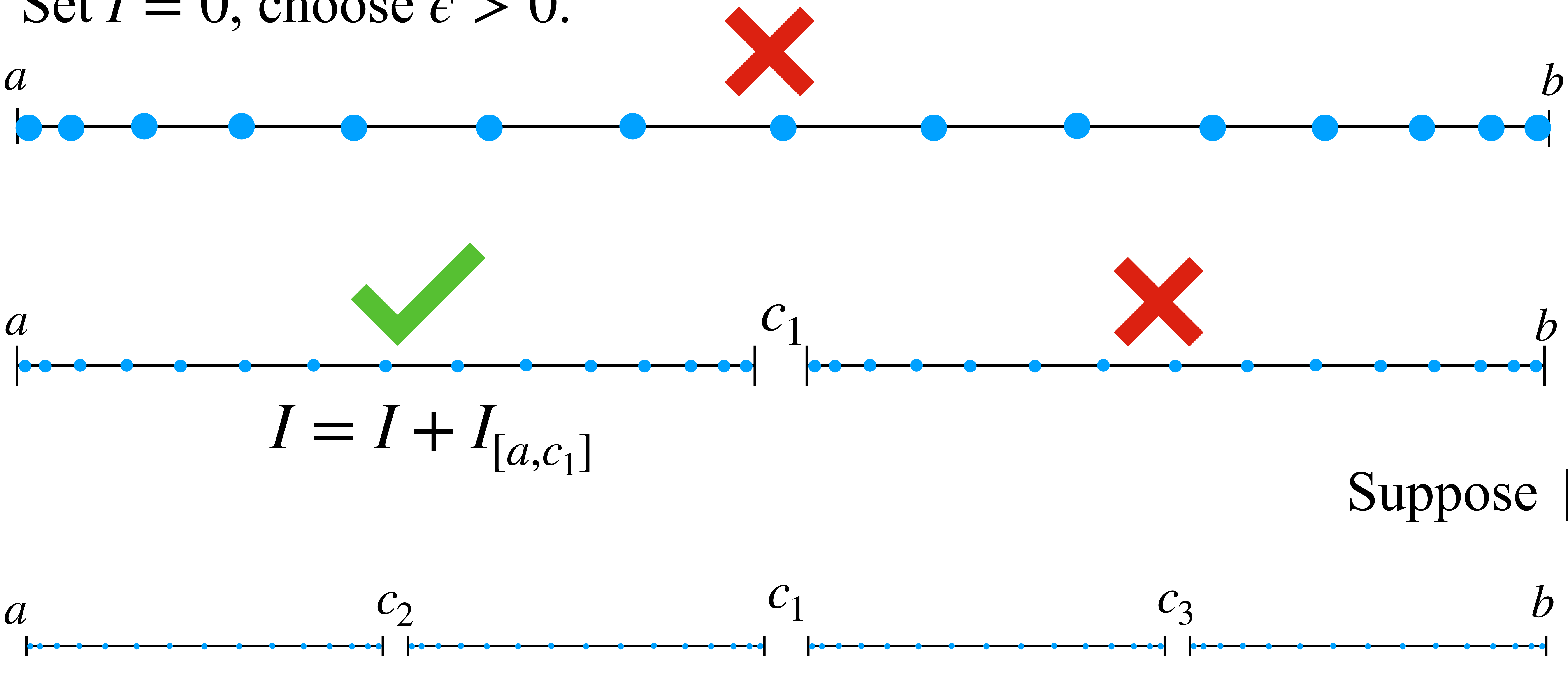
$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Notation:

$$[f] = \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix}$$

Implementing the Adaptive Algorithm

Set $I = 0$, choose $\epsilon > 0$.



$\longrightarrow I_{[a,b]}$

$\longrightarrow I_{[a,c_1]} + I_{[c_1,b]}$

$$I = I + I_{[a,c_1]}$$

Suppose $|I_{[a,b]} - I_{[a,c_1]} - I_{c_1,b]}| \geq \epsilon$

$\longrightarrow$

$I_{[a,c_2]} + I_{[c_2,c_1]}$

$I_{[a,c_2]} + I_{[c_2,c_1]}$

Suppose $|I_{[a,c_1]} - I_{[a,c_2]} - I_{c_2,c_1]}| < \epsilon$

Suppose $|I_{[c_1,b]} - I_{[c_1,c_3]} - I_{c_3,b]}| \geq \epsilon$

Goal: Evaluate the following;

$$I = \int_a^b f(x)\ \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Notation:

$$[f] = \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix}$$

Estimate on $[a_0, b_0] \subset [a, b]$:

$$I_{a_0 b_0} = p(x_{k,k})\exp(i(g(x_{k,k})) - p(x_{1,k})\exp(ig(x_{1,k}))$$

Numerical Experiments

$$I_1(x) = \int_0^1 \exp(i\lambda x^2)\exp(-x)x dx$$

$$f(x) = \exp(-x)x$$

$$g(x) = \lambda x^2$$

$$I_2(x) = \int_0^1 \exp(i\lambda x^4)\frac{1}{0.01 + x^4} dx$$

$$f(x) = \frac{1}{0.01 + x^4}$$

$$g(x) = \lambda x^4$$

Integral	Range of λ	Avg Time Adap Levin	Avg Time Adap Gauss	Ratio	Max Observed Difference
I_1	$10^0 - 10^1$	4.83×10^{-05}	2.23×10^{-06}	0.05	9.94×10^{-13}
	$10^1 - 10^2$	9.16×10^{-05}	5.93×10^{-06}	0.06	1.32×10^{-12}
	$10^2 - 10^3$	1.23×10^{-04}	4.71×10^{-05}	0.38	1.01×10^{-12}
	$10^3 - 10^4$	1.58×10^{-04}	4.15×10^{-04}	2.64	7.53×10^{-13}
	$10^4 - 10^5$	2.02×10^{-04}	4.26×10^{-03}	21.12	9.99×10^{-13}
	$10^5 - 10^6$	2.29×10^{-04}	3.99×10^{-02}	173.87	1.00×10^{-12}
	$10^6 - 10^7$	2.51×10^{-04}	3.71×10^{-01}	1476.94	4.00×10^{-13}
I_2	$10^0 - 10^1$	1.38×10^{-04}	2.29×10^{-06}	0.02	1.94×10^{-12}
	$10^1 - 10^2$	2.88×10^{-04}	1.38×10^{-05}	0.05	1.97×10^{-12}
	$10^2 - 10^3$	4.17×10^{-04}	1.07×10^{-04}	0.26	3.58×10^{-12}
	$10^3 - 10^4$	4.74×10^{-04}	9.00×10^{-04}	1.90	3.32×10^{-12}
	$10^4 - 10^5$	5.25×10^{-04}	8.84×10^{-03}	16.85	2.20×10^{-12}
	$10^5 - 10^6$	5.77×10^{-04}	8.05×10^{-02}	139.52	3.53×10^{-12}
	$10^6 - 10^7$	6.44×10^{-04}	8.01×10^{-01}	1243.52	2.57×10^{-12}

Source: [1]

Goal: Evaluate the following;

$$I = \int_a^b f(x) \exp(ig(x))dx$$

First: Find $p(x)$ s.t

$$f(x) = p'(x) + p(x) \cdot ig'(x)$$

Notation:

$$[f] = \begin{pmatrix} f(x_{1,k}) \\ \vdots \\ f(x_{k,k}) \end{pmatrix}$$

Estimate on $[a_0, b_0] \subset [a, b]$:

$$I_{a_0 b_0} = p(x_{k,k})\exp(i(g(x_{k,k})) - p(x_{1,k})\exp(ig(x_{1,k}))$$

Two-Dimensional Levin Method

Goal: Generalize to Two Dimensions

$$I = \iint_R f(x, y)\exp(i\vec{\varpi} \cdot \vec{g}(x, y))dxdy \tag{1}$$

1. $\vec{g} : \mathbb{R}^2 \to \mathbb{R}^2,$
2. $f : \mathbb{R}^2 \to \mathbb{R},$
3. $\vec{\varpi} \in \mathbb{R}^2,$
4. $R = [a, b] \times [c, d] \subset \mathbb{R}^2$

$$\mathfrak{Re}\{I\} = \iint_R f(x, y)\cos(\vec{\varpi} \cdot \vec{g}(x, y))dxdy$$

$$\mathfrak{Im}\{I\} = \iint_R f(x, y)\sin(\vec{\varpi} \cdot \vec{g}(x, y))dxdy$$

Method: Find an antiderivative of the integrand in Eq.(1).

Goal: Generalize to Two Dimensions

$$I = \iint_R f(x, y) \exp(i\overrightarrow{\omega} \cdot \overrightarrow{g}(x, y)) dx dy \tag{1}$$

Search for a function $\overrightarrow{p} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x, y)\ \exp(i\overrightarrow{\omega} \cdot \overrightarrow{g}(x, y)) = \nabla \cdot \left(\overrightarrow{p}(x, y)\ \exp(i\overrightarrow{\omega} \cdot \overrightarrow{g}(x, y))\right)$$

Plugging this into I and applying the divergence theorem yields

$$I = \iint_R \nabla \cdot (\overrightarrow{p}(x, y)\ \exp(i\overrightarrow{\omega} \cdot \overrightarrow{g}(x, y)) dx dy = \int_{\partial R} \overrightarrow{p}(x, y)\ \exp(i\overrightarrow{\omega} \cdot \overrightarrow{g}(x, y)) dx dy$$

Goal: Generalize to Two Dimensions

$$I = \iint_R f(x,y)\exp(i\vec{\omega}\cdot\vec{g}(x,y))dxdy \tag{1}$$

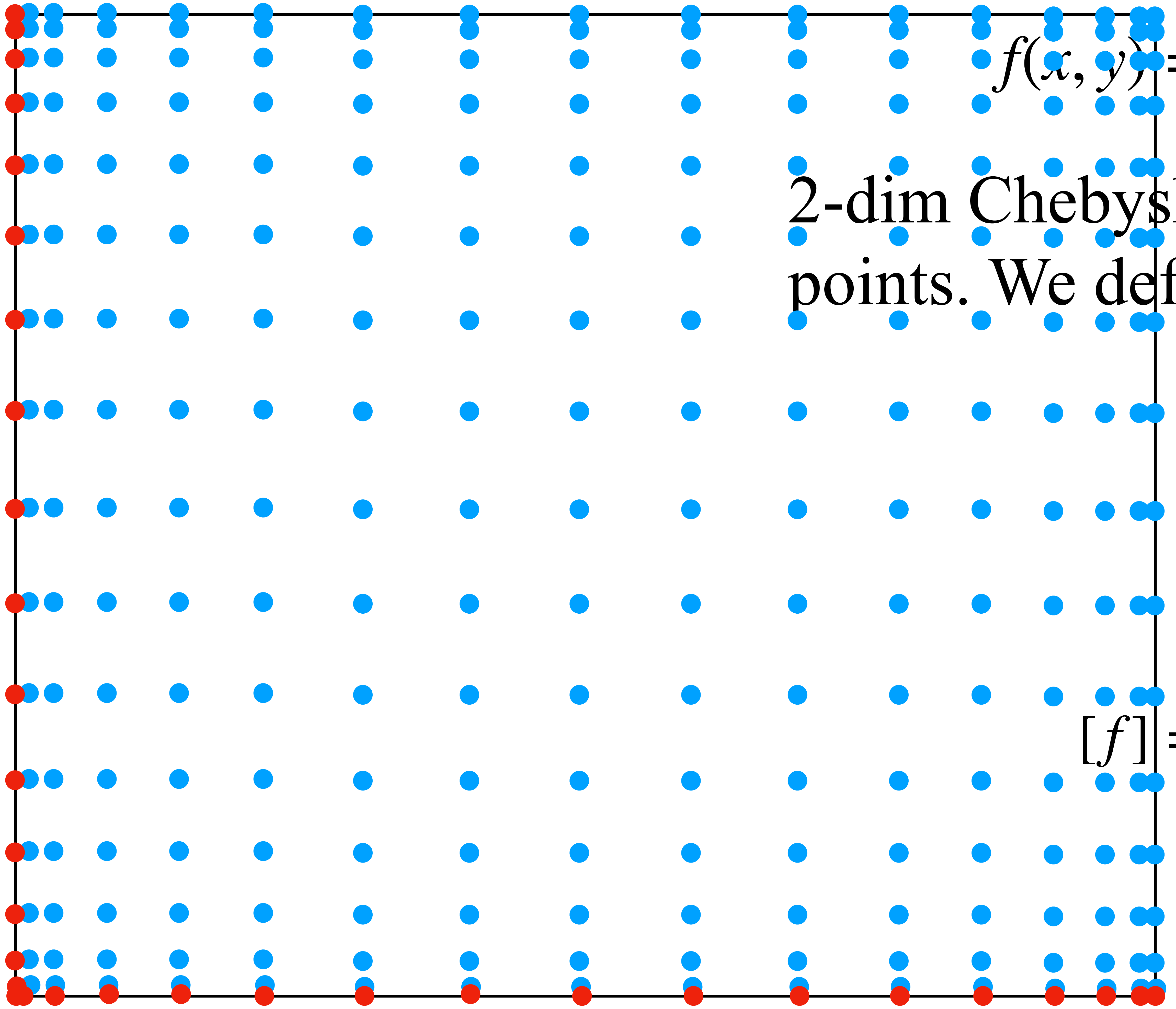
Search for a function $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x,y)\ \exp(i\vec{\omega}\cdot\vec{g}(x,y)) = \nabla\cdot\left(\vec{p}(x,y)\ \exp(i\vec{\omega}\cdot\vec{g}(x,y))\right)$$

$$= \left(\nabla\cdot\vec{p}(x,y) + i\omega^t D\vec{g}(x,y)\vec{p}(x,y)\right)\exp(i\vec{\omega}\cdot\vec{g}(x,y))$$

$$\boxed{\implies f(x,y) = \nabla\cdot\vec{p}(x,y) + i\omega^t D\vec{g}(x,y)\vec{p}(x,y)}$$

Method of Discretization: Two Dimensional Chebyshev Interpolation



Approximate a function f as,

$$f(x, y) = \sum_{i,j=0}^{n-1} a_{ij} T_i(x) T_j(y)$$

2-dim Chebyshev grid containing n^2 points. We define $[f]$ as;

$$[f] = \begin{pmatrix} f(x_{1,n}, y_{1,n}) \\ f(x_{2,n}, y_{1,n}) \\ \vdots \\ f(x_{n,n}, y_{1,n}) \\ f(x_{1,n}, y_{2,n}) \\ \vdots \\ f(x_{n,n}, y_{2,n}) \\ \vdots \\ f(x_{n,n}, y_{n,n}) \end{pmatrix}$$

Method of Discretization: Two Dimensional Chebyshev Interpolation

Consider the special differentiation matrix $\mathscr{D}$ from the 1-dimensional discretization.

$$I \otimes \mathscr{D}[f] = \left[\frac{\partial f}{\partial x}\right] \qquad \mathscr{D} \otimes I[f] = \left[\frac{\partial f}{\partial y}\right] \qquad A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}$$

The matrix $\mathscr{A}$ is defined as

$$\mathscr{A} = \begin{pmatrix} I \otimes \mathscr{D} & \mathscr{D} \otimes I \end{pmatrix} + \begin{pmatrix} \text{diag}(\omega_1) & \text{diag}(\omega_2 \end{pmatrix} \begin{pmatrix} \text{diag}\left[\frac{\partial g_1}{\partial x_1}\right] & \text{diag}\left[\frac{\partial g_1}{\partial x_2}\right] \\ \text{diag}\left[\frac{\partial g_2}{\partial x_1}\right] & \text{diag}\left[\frac{\partial g_2}{\partial x_2}\right] \end{pmatrix}$$

$$\mathscr{A} \begin{pmatrix} [p_1] \\ [p_2] \end{pmatrix} = [f]$$

Method of Discretization: Two Dimensional Chebyshev Interpolation

$$\mathscr{A}\begin{pmatrix}[p_1]\\ [p_2]\end{pmatrix} = [f] \longrightarrow \begin{pmatrix}[p_1]\\ [p_2]\end{pmatrix} = \mathscr{A}^{-1}[f]$$

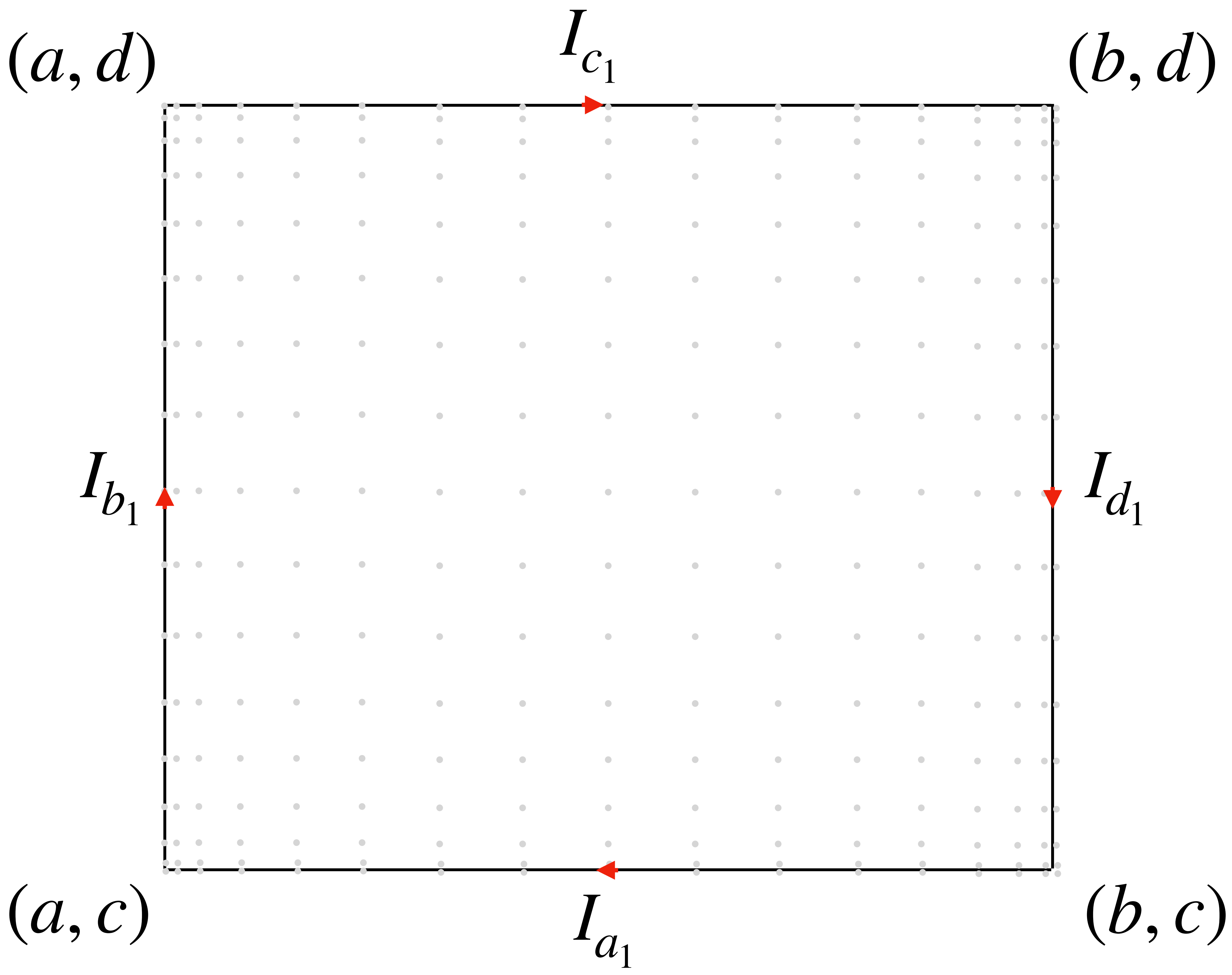
We obtain the values of p_1 and p_2 at the 2-dimensional Chebyshev quadrature, again numerically inverting $\mathscr{A}$ via QR or SVD.

Given a sub-rectangle $R_0 \subset R$ and associated nodes $\{x_{i,k}, y_{j,k}\}$, we have

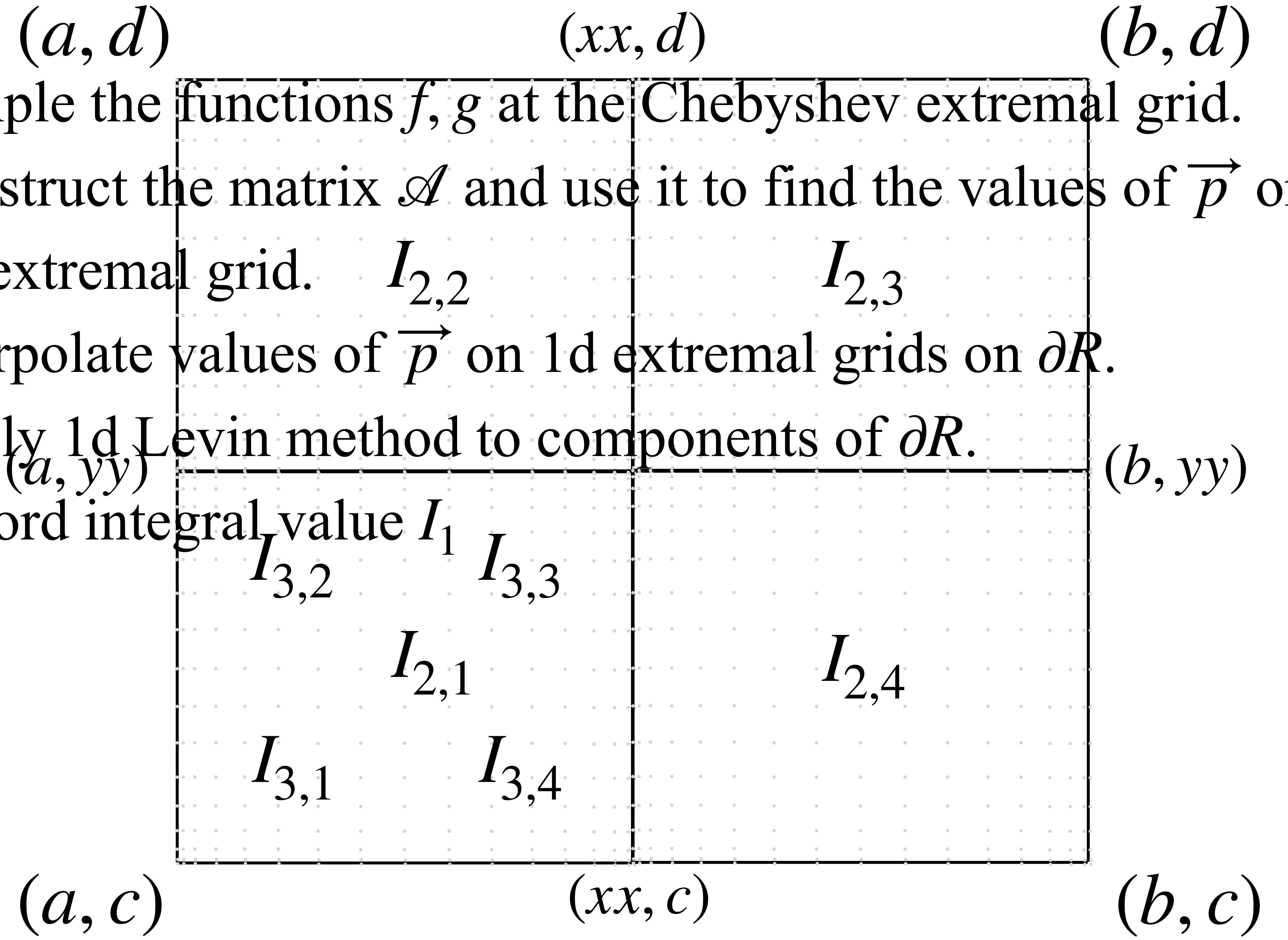
$$\iint_{R_0} f(x,y)\exp(i\overrightarrow{\omega}\cdot\overrightarrow{g}(x,y))dxdy = \int_{\partial R_0} \overrightarrow{p}(x,y)\exp(i\overrightarrow{\omega}\cdot\overrightarrow{g}(x,y))dxdy$$

∂R_0 is the union of four lines, the integral over each of which is approximated via 1d method.

Implementing the Adaptive Algorithm



- Sample the functions f, g at the Chebyshev extremal grid.
- Construct the matrix $\mathscr{A}$ and use it to find the values of $\overrightarrow{p}$ on the extremal grid.
- Interpolate values of $\overrightarrow{p}$ on 1d extremal grids on ∂R.
- Apply 1d Levin method to components of ∂R.
- Record integral value I_1



$$I_1 = I_{a_1} + I_{b_1} + I_{c_1} + I_{d_1}$$

$$\boldsymbol{I_2} = \boldsymbol{I_{2,1}} + \boldsymbol{I_{2,2}} + \boldsymbol{I_{2,3}} + \boldsymbol{I_{2,4}}$$

$$\boldsymbol{I_3} = \boldsymbol{I_{3,1}} + \boldsymbol{I_{3,2}} + \boldsymbol{I_{3,3}} + \boldsymbol{I_{3,4}}$$

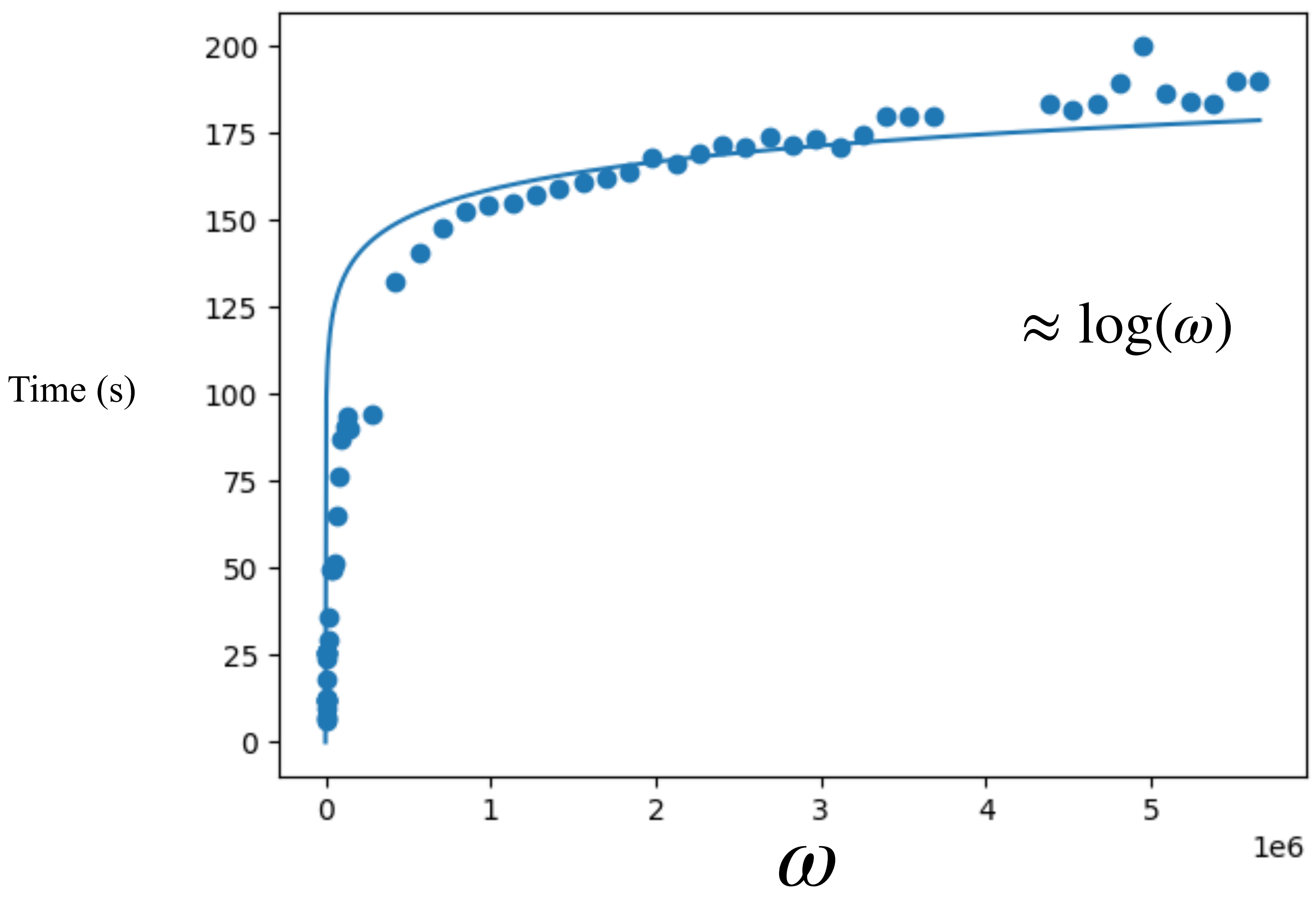
$$|I_1 - I_2| \geq \epsilon$$

$$|I_{2,1} - I_3| \geq \epsilon$$

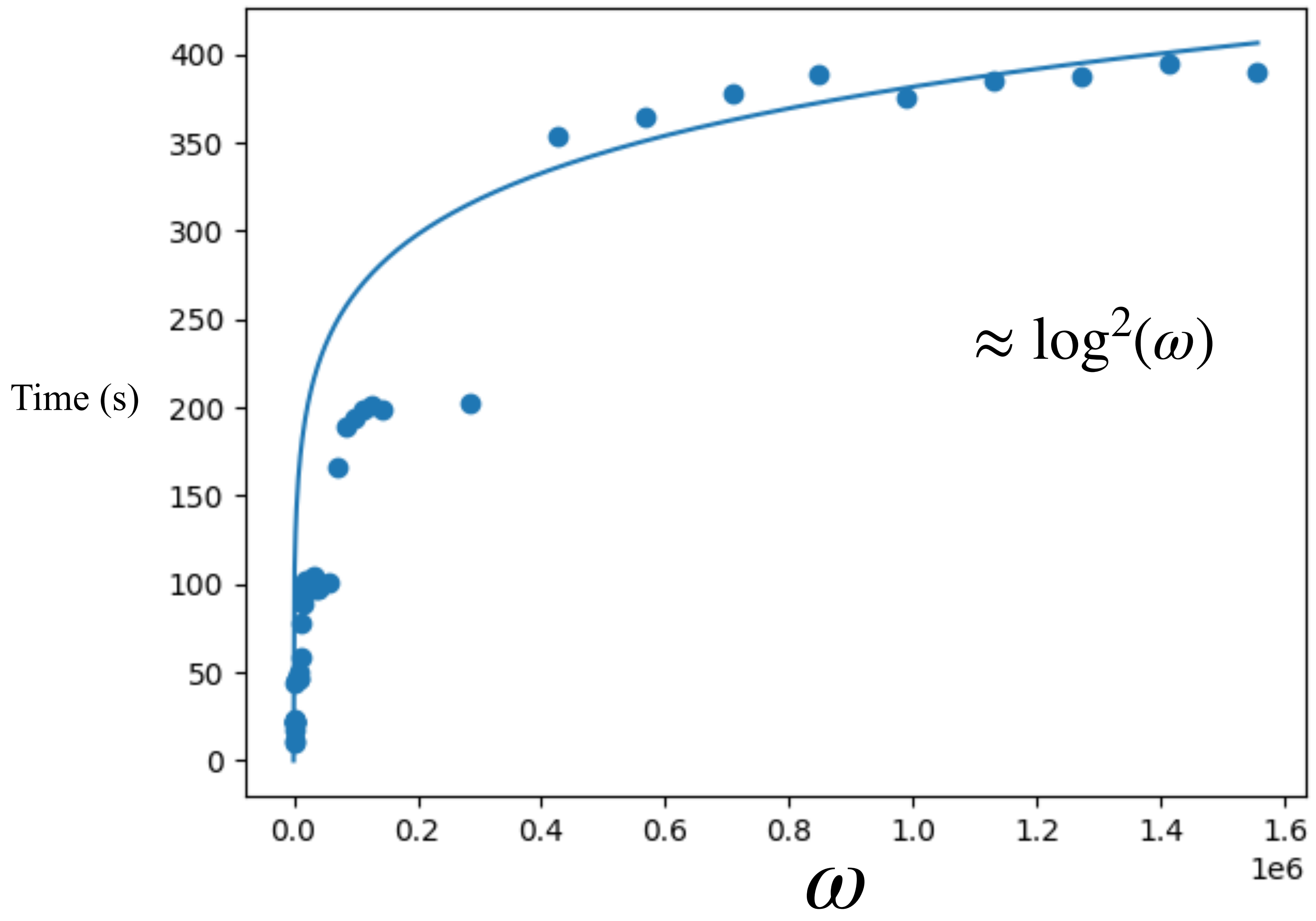
Closing Remarks

I am currently working on speeding up the algorithm

↳ Cases where Dg reaches a singularity, computation time is length



$$Dg(x', y') = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$$



$$Dg(x', y') = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

References

[1] Shukui Chen, Kirill Serkh, James Bremer. *The Adaptive Levin Method*

[2] David Levin. *Procedures for Computing One- and Two-Dimensional Integrals of Functions with Rapid Irregular Oscillations*